Nominal String Diagrams

Samuel Balco
Department of Informatics,University of Leicester, United Kingdom
https://gdlyrttnap.pl
sb782@leicester.ac.uk

Alexander Kurz
Department of Computer Science, Chapman University, Orange California, USA
akurz@chapman.edu

Abstract

We introduce nominal string diagrams as string diagrams internal in the category of nominal sets. This requires us to take nominal sets as a monoidal category, not with the cartesian product, but with the separated product. To this end, we develop the beginnings of a theory of monoidal categories internal in a symmetric monoidal category. As an instance, we obtain a notion of a nominal PROP as a PROP internal in nominal sets. A 2-dimensional calculus of simultaneous substitutions is an application.

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1 Introduction

One reason for the success of string diagrams, see [19] for an overview, can be formulated by the slogan 'only connectivity matters' [4, Sec.10.1]. Technically, this is usually achieved by ordering input and output wires and using their ordinal numbers as implicit names. We write $\underline{n} = \{1, \dots n\}$ to denote the set of n numbered wires and $f : \underline{n} \to \underline{m}$ for diagrams f with n inputs and m outputs. The approach of using order to implicitly name wires is particularly convenient for the generalisations of Lawvere theories known as PROPs [16]. In particular, the paper on composing PROPs [13] has been influential [2, 3].

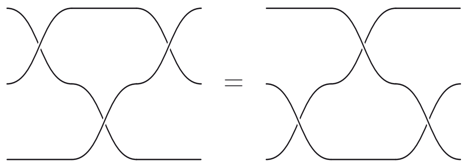
On the other hand, if only connectivity matters, it is natural to consider a formalisation of PROPs in which wires are not ordered. Thus, instead of ordering wires, we fix a countably infinite set $\mathcal{N}$ of 'names' $a, b, \dots$, on which the only supported operation or relation is equality. Mathematically, this means that we work internally in the category of nominal sets introduced by Gabbay and Pitts [8, 18]. In the remainder of the introduction, we highlight some of the features of this approach.

Partial commutative vs total symmetric tensor. One reason why ordered names are convenient is that the tensor $\oplus$ is given by the categorical coproduct (addition) in the skeleton $\mathbb{F}$ of the category of finite sets. Even though $\underline{n} \oplus \underline{m} = \underline{m} \oplus \underline{n}$ on objects, the tensor is not commutative but only symmetric, since the canonical arrow $\underline{n} \oplus \underline{m} \to \underline{m} \oplus \underline{n}$ is not the identity.

On the other hand, in the category $\mathsf{n}\mathbb{F}$ of finite subsets of $\mathcal{N}$ (which is equivalent to $\mathbb{F}$ as an ordinary category), there is a commutative tensor $A \uplus B$ given by union of disjoint sets. The feature that makes commutativity possible is that $\uplus$ is partial with $A \uplus B$ defined if and only if $A \cap B = \emptyset$.

While it would be interesting to develop a general theory of partially monoidal categories, our approach in this paper is based on the observation that the partial operation $\uplus : \mathsf{n}\mathbb{F} \times \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$ is a total operation $\uplus : \mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$ where $*$ is the separated product of nominal sets [18].

Symmetries disappear in 3 dimensions. From a graphical point of view, the move from ordered wires to named wires corresponds to moving from planar graphs to graphs in 3 dimensions. Instead of having a one dimensional line of inputs or outputs, wires are now sticking out of a plane [12]. As a benefit there are no wire-crossings, or, more technically, there are no symmetries to take care of. This simplifies the rewrite rules of calculi formulated in the named setting. For example, rules such as



are not needed anymore. For more on this compare Figs 3 and 4.

Example: Simultaneous Substitutions. Substitutions $[a{\mapsto}b]$ can be composed sequentially and in parallel as in

$$[a{\mapsto}b]\,;[b{\mapsto}c] = [a{\mapsto}c] \qquad\qquad [a{\mapsto}b] \uplus [c{\mapsto}d] = [a{\mapsto}b, c{\mapsto}d].$$

We call $\uplus$ the tensor, or the monoidal or vertical or parallel composition. Semantically, the simultaneous substitution on the right-hand side above, will correspond to the function $f : \{a, c\} \to \{b, d\}$ satisfying $f(a) = b$ and $f(c) = d$. Importantly, parallel composition of simultaneous substitutions is partial. For example, $[a{\mapsto}b] \uplus [a{\mapsto}c]$ is undefined, since there is no function $\{a\} \to \{b, c\}$ that maps a simultaneously to both b and c.

The advantages of a 2-dimensional calculus for simultaneous substitutions over a 1-dimensional calculus are the following. A calculus of substitutions is an algebraic representation, up to isomorphism, of the category $\mathsf{n}\mathbb{F}$ of finite subsets of $\mathcal{N}$. In a 1-dimensional calculus, operations $[a{\mapsto}b]$ have to be indexed by finite sets S

$$[a{\mapsto}b]_S : S \cup \{a\} \to S \cup \{b\}$$

for sets S with $a, b \notin S$. On the other hand, in a 2-dimensional calculus with an explicit operation $\uplus$ for set union, indexing with subsets S is unnecessary. Moreover, while the swapping

$$\{a, b\} \to \{a, b\}$$

in the 1-dimensional calculus needs an auxiliary name such as c in $[a{\mapsto}c]_{\{b\}}\,;[b{\mapsto}a]_{\{c\}}\,;[c{\mapsto}a]_{\{b\}}$ it is represented in the 2-dimensional calculus directly by

$$[a{\mapsto}b] \uplus [b{\mapsto}a]$$

Finally, while it is possible to write down the equations and rewrite rules for the 1-dimensional calculus, it does not appear as particularly natural. In particular, only in the 2-dimensional calculus, will the swapping have a simple normal form such as $[a{\mapsto}b] \uplus [b{\mapsto}a]$ (unique up to commutativity of $\uplus$).

Overview. In order to account for partial tensors, Section 3 develops the notion of a monoidal category internal in a monoidal category. Section 4 is devoted to examples, while Section 5 introduces the notion of a nominal PROP and Section 6 shows that the categories of ordinary and of nominal PROPs are equivalent.

2 Setting the Scene: String Diagrams and Nominal Sets

We review some of the terminology but need to refer to the literature for details.

2.1 String Diagrams and PROPs

String diagrams are a 2-(or higher)-dimensional notation for monoidal categories [12]. Their algebraic theory can be formalised by PROPs as defined by MacLane [15]. There is also the weaker notion by Lack [13], see Remark 2.9 of Zanasi [22] for a discussion.

A PROP (*products and permutation category*) is a symmetric strict monoidal category, with natural numbers as objects, where the monoidal tensor $\oplus$ is addition. Moreover, PROPs, along with strict symmetric monoidal functors, that are identities on objects, form the category PROP. A PROP contains all bijections between numbers as they can be be generated from the symmetry (twist) $\sigma : 1 \oplus 1 \to 1 \oplus 1$ and from the parallel composition $\oplus$ and sequential composition ; (which we write in diagrammatic order). We denote by $\sigma_{n,m}$ the canonical symmetry $n \oplus m \to m \oplus n$. Functors between PROPs preserve bijections.

PROPs can be presented in algebraic form by operations and equations as *symmetric monoidal theories* (SMTs) [22].

An SMT (Σ, E) has a set Σ of generators, where each generator $\gamma \in \Sigma$ is given an arity m and co-arity n, usually written as $\gamma : m \to n$ and a set E of equations, which are pairs of Σ-terms. Σ-terms can be obtained by composing generators in Σ with the unit $id : 1 \to 1$ and symmetry $\sigma : 2 \to 2$, using either the parallel or sequential composition (see Fig 1). Equations E are pairs of Σ-terms with the same arity and co-arity.

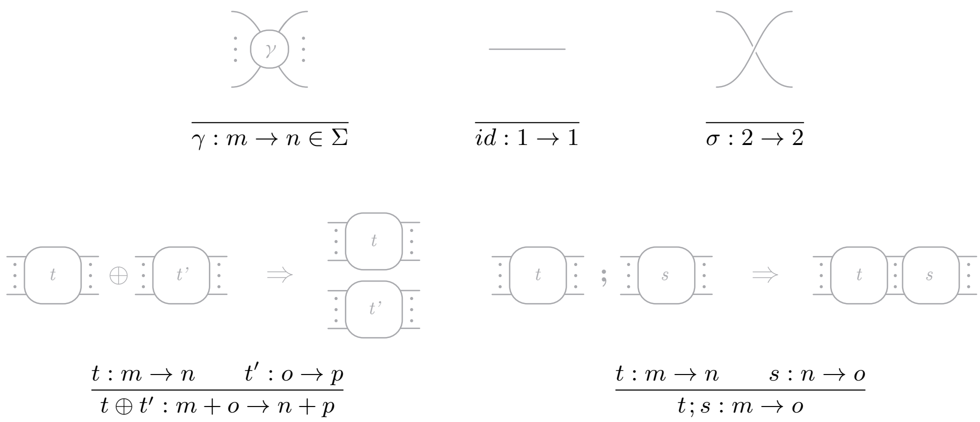


Figure 1 SMT Terms

Given an SMT $\langle \Sigma, E \rangle$, we can freely generate a PROP, by taking Σ-terms as arrows, modulo

- the equations stating that, together with id, the compositions ; and $\oplus$ form monoids
- the equations of Fig 2
- the equations E

$$\sigma_{1,1} ; \sigma_{1,1} = id_2 \quad \text{(SMT-sym)}$$
$$(s ; t) \oplus (u ; v) = (s \oplus u) ; (t \oplus v) \quad \text{(SMT-ch)}$$
$$(t \oplus id_z) ; \sigma_{n,z} = \sigma_{m,z} ; (id_z \oplus t) \quad \text{(SMT-nat)}$$

Figure 2 Equations of symmetric monoidal categories

PROPs have a nice 2-dimensional notation, where sequential composition is horizontal composition of diagrams, and parallel/tensor composition is vertical stacking of diagrams (see Fig 1). We now present the SMTs of bijections $\mathbb{B}$, injections $\mathbb{I}$, surjections $\mathbb{S}$, functions $\mathbb{F}$, partial functions $\mathbb{P}$, relations $\mathbb{R}$ and monotone maps $\mathbb{M}$.[1] The diagram in Fig 3 shows the generators and the equations that need to be added to the empty SMT, to get a presentation of the given theory. To ease comparison with the corresponding nominal monoidal theories in Fig 4 later we also added on a striped background the equations for wire-crossings that are already implied by the naturality of symmetries (SMT-nat). These are equations that are part of the definition of a PROP in the sense of MacLane [15] but not in the sense of Lack [13]. The right-hand equation for bijections $\mathbb{B}$ is (SMT-sym) and holds in all symmetric monoidal theories. We list it here to emphasise the difference with Fig 4.

2.2 Nominal Sets

Let $\mathcal{N}$ be a countably infinite set of 'names' or 'atoms'. Let $\mathfrak{S}$ be the group of finite[2] permutations $\mathcal{N} \to \mathcal{N}$. An element $x \in X$ of a group action $\mathfrak{S} \times X \to X$ is supported by $S \subseteq \mathcal{N}$ if $\pi \cdot x = x$ for all $\pi \in \mathfrak{S}$ such that π restricted to S is the identity. A group action $\mathfrak{S} \times X \to X$ such that all elements of X have finite support is called a *nominal set*. We write $\mathsf{supp}(x)$ for the minimal support of x and say that a is fresh for x if $a \notin \mathsf{supp}(x)$. Nom for the category of nominal sets, which has as maps the *equivariant* functions, that is, those functions that respect the permutation action. Our main example is the category of simultaneous substitutions:

▶ **Example 1** ($\mathsf{n}\mathbb{F}$). We denote by $\mathsf{n}\mathbb{F}$ the category of finite subsets of $\mathcal{N}$ with all functions. While $\mathsf{n}\mathbb{F}$ is a category, it also carries additional nominal structure. In particular, both the set of objects and the set of arrows are nominal sets with $\mathsf{supp}(A) = A$ and $\mathsf{supp}(f) = A \cup B$ for $f : A \to B$. The categories of injections, surjections, bijections, partial functions and relations are further examples along the same lines.

[1] The theory of monotone maps $\mathbb{M}$ does not include equations involving the symmetry σ and is in fact presented by a so-called PRO rather than a PROP. However, in this paper we will only be dealing with theories presented by PROPs (the reason why this is the case is illustrated in the proof of Proposition 20).

[2] A permutation is called finite if it is generated by finitely many transpositions.

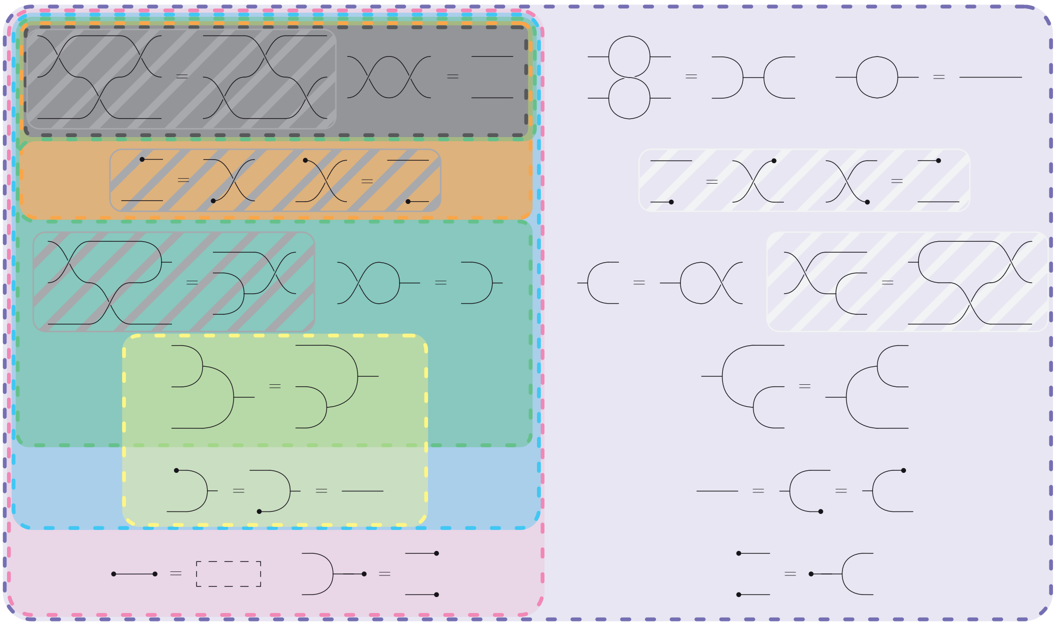


Figure 3 Symmetric monoidal theories (compiled from [14])

3 Internal monoidal categories

We introduce the notion of an internal monoidal category. Given a symmetric monoidal category $(\mathcal{V}, I, \otimes)$ with finite limits, we are interested in categories $\mathbb{C}$, internal in $\mathcal{V}$, that carry a monoidal structure not of type $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ but of type $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$. This will allow us to account for the partiality of $\uplus$ discussed in the introduction. We present our motivating example before we give Definition 11.

▶ **Example 2.**

- The symmetric monoidal (closed) category $(\mathsf{Nom}, 1, *)$ of nominal sets with the separated product $*$ is defined as follows [18]. 1 is the terminal object, i.e. a singleton with empty support. The separated product of two nominal sets is defined as $A * B = \{(a, b) \in A \times B \mid \mathsf{supp}(a) \cap \mathsf{supp}(b) = \emptyset\}$.
- The category $\mathsf{n}\mathbb{F}$ of Example 1 is an internal monoidal category with monoidal operation given by $A \uplus B = A \cup B$ if A, B are disjoint and $f \uplus f' = f \cup f'$ if A, A' and B, B' are disjoint where $f : A \to B$ and $f' : A' \to B'$.

$(\mathsf{n}\mathbb{F}, \emptyset, \uplus)$ as defined in the previous example is not a monoidal category, since $\uplus$, being partial, is not an operation of type $\mathsf{n}\mathbb{F} \times \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}$. The purpose of this section is to define the notion of internal monoidal category and to show that $(\mathsf{n}\mathbb{F}, \emptyset, \uplus)$ is an internal monoidal category in $(\mathsf{Nom}, 1, *)$ with $\uplus$ of type

$$\uplus : \mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F} \to \mathsf{n}\mathbb{F}.$$

To this end we need to extend $* : \mathsf{Nom} \times \mathsf{Nom} \to \mathsf{Nom}$ to

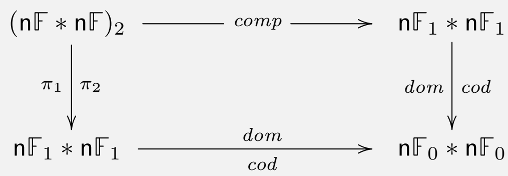
$$* : \mathsf{Cat}(\mathsf{Nom}) \times \mathsf{Cat}(\mathsf{Nom}) \to \mathsf{Cat}(\mathsf{Nom})$$

where we denote by $\mathsf{Cat}(\mathsf{Nom})$, the category of (small) internal categories in Nom. The necessary (and standard) notation from internal categories is reviewed in Appendix A.

▶ Remark 3. Let $\mathbb{C}$ be an internal category in a symmetric monoidal category $(\mathcal{V}, I, \otimes)$ with finite limits. Since $\otimes$ need not preserve finite limits, we cannot expect that defining $(\mathbb{C} \otimes \mathbb{C})_0 = \mathbb{C}_0 \otimes \mathbb{C}_0$ and $(\mathbb{C} \otimes \mathbb{C})_1 = \mathbb{C}_1 \otimes \mathbb{C}_1$ results in $\mathbb{C} \otimes \mathbb{C}$ being an internal category.

Consequently, putting $(\mathbb{C} \otimes \mathbb{C})_1 = \mathbb{C}_1 \otimes \mathbb{C}_1$ does not extend $\otimes$ to an operation $\mathsf{Cat}(\mathcal{V}) \times \mathsf{Cat}(\mathcal{V}) \to \mathsf{Cat}(\mathcal{V})$. To show what goes wrong in a concrete instance is the purpose of the next example.

▶ **Example 4.** Define a binary operation $\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F}$ as $(\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F})_0 = \mathsf{n}\mathbb{F}_0 * \mathsf{n}\mathbb{F}_0$ and $(\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F})_1 = \mathsf{n}\mathbb{F}_1 * \mathsf{n}\mathbb{F}_1$. Then $\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F}$ cannot be equipped with the structure of an internal category. Indeed, assume for a contradiction that there was an appropriate pullback $(\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F})_2$ and arrow *comp* such that the two diagrams commute:



Let $\delta_{xy} : \{x\} \to \{y\}$ be the unique function in $\mathsf{n}\mathbb{F}$ of type $\{x\} \to \{y\}$. Then $((\delta_{ac}, \delta_{bd}), (\delta_{cb}, \delta_{da}))$, which can be depicted as

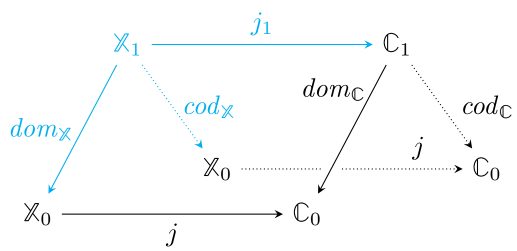
$$\begin{array}{ccccc} \{a\} & \xrightarrow{\delta_{ac}} & \{c\} & \xrightarrow{\delta_{cb}} & \{b\} \\ \{b\} & \xrightarrow[\delta_{bd}]{} & \{d\} & \xrightarrow[\delta_{da}]{} & \{a\} \end{array}$$

is in the pullback $(\mathsf{n}\mathbb{F} * \mathsf{n}\mathbb{F})_2$, but there is no *comp* such that the two squares above commute, since $comp((\delta_{ac}, \delta_{bd}), (\delta_{cb}, \delta_{da}))$ would have to be $(\delta_{ab}, \delta_{ba})$, which do not have disjoint support and therefore are not in $\mathsf{n}\mathbb{F}_1 * \mathsf{n}\mathbb{F}_1$. ◀

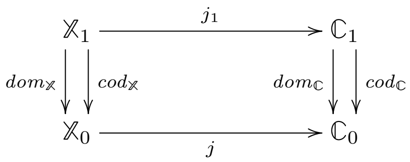
The solution to the problem consists in assuming that the given symmetric monoidal category with finite limits $(\mathcal{V}, 1, \otimes)$ is semi-cartesian (aka affine), that is, the unit 1 is the terminal object. In such a category there are canonical arrows natural in A and B

$$j : A \otimes B \to A \times B$$

and we can use them to define arrows $j_1 : (\mathbb{C} \otimes \mathbb{C})_1 \to \mathbb{C}_1 \times \mathbb{C}_1$ that give us the right notion of tensor on arrows. From our example $\mathsf{n}\mathbb{F}$ above, we know that we want arrows (f, g) to be in $(\mathbb{C} \otimes \mathbb{C})_1$ if $dom(f) \cap dom(g) = \emptyset$ and $cod(f) \cap cod(g) = \emptyset$. We now turn this into a category theoretic definition, which, in fact, is an instace of the general and well-known construction of pulling back an internal category $\mathbb{C}$ along an arrow $j : X \to \mathbb{C}_0$ to yield an internal category $\mathbb{X}$ with $\mathbb{X}_0 = X$ and $\mathbb{X}_1$ the pullback of $\langle dom_{\mathbb{C}}, cod_{\mathbb{C}} \rangle$ along $j \times j$, or, equivalently, the limit in the following diagram

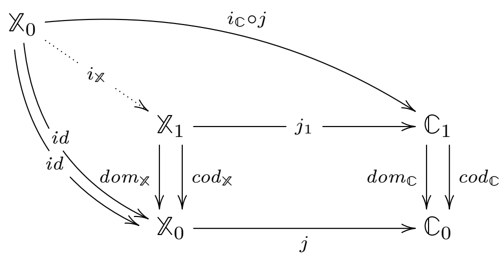


which we abbreviate to



(1)

Next we define $i : \mathbb{X}_0 \to \mathbb{X}_1$ as the arrow into the limit $\mathbb{X}_1$ given by

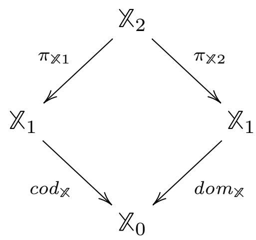


(2)

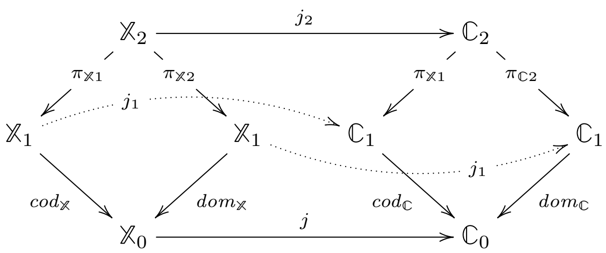
from which one reads off

$$dom_{\mathbb{X}} \circ i_{\mathbb{X}} = id_{\mathbb{X}_0} = cod_{\mathbb{X}} \circ i_{\mathbb{X}}$$

Next, $\mathbb{X}_2$ is the pullback

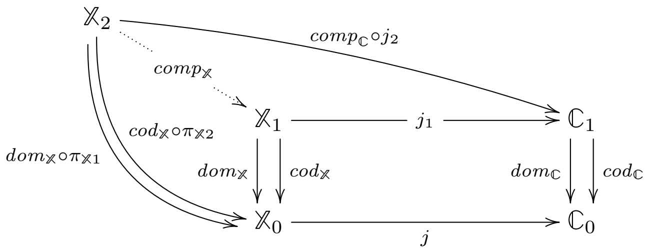


Recalling the definition of j_1 from (1), there is also a corresponding $j_2 : \mathbb{X}_2 \to \mathbb{C}_2$ due to the fact that the product of pullbacks is a pullback of products.



(3)

Recall the definition of the limit $\mathbb{X}_1$ from (1). Then $comp_{\mathbb{X}} : \mathbb{X}_2 \to \mathbb{X}_1$ is the arrow into $\mathbb{X}_1$



(4)

from which one reads off

$$dom_{\mathbb{X}} \circ comp_{\mathbb{X}} = dom_{\mathbb{X}} \circ \pi_{\mathbb{X}1} \qquad cod_{\mathbb{X}} \circ comp_{\mathbb{X}} = cod_{\mathbb{X}} \circ \pi_{\mathbb{X}2} \qquad j_1 \circ comp_{\mathbb{X}} = comp_{\mathbb{C}} \circ j_2$$

and the remaining equations $comp_{\mathbb{X}} \circ \langle i_{\mathbb{X}} \circ dom_{\mathbb{X}}, id_{\mathbb{X}_1} \rangle = id_{\mathbb{X}_1} = comp_{\mathbb{X}} \circ \langle id_{\mathbb{X}_1}, i_{\mathbb{X}} \circ cod_{\mathbb{X}} \rangle$ are also not difficult to prove.

We have seen that the pullback of an internal category $\mathbb{C}$ along an arrow j with codomain $\mathbb{C}_0$ is an internal category:

▶ **Proposition 5.** Given an internal category $\mathbb{C}$ and an arrow $j : X \to \mathbb{C}_0$ there is an internal category $\mathbb{X}$ and an internal functor $\mathbb{j} : \mathbb{X} \to \mathbb{C}$ such that $\mathbb{X}_0 = X$ and $\mathbb{j}_0 = j$.

Moreover, this internal category $\mathbb{X}$, or rather $\mathbb{j} : \mathbb{X} \to \mathbb{C}$, has a universal property known as a cartesian lifting. To make this precise, we recall the notion of a fibred category, or fibration.

▶ **Definition 6** (Fibration [11, 20]). If $P : \mathcal{W} \to \mathcal{V}$ is a functor, then $\mathbb{j} : \mathbb{X} \to \mathbb{C}$ is a cartesian lifting of $j : X \to P\mathbb{C}$ if for all $\mathbb{k} : \mathbb{W} \to \mathbb{C}$ and all $h : P\mathbb{W} \to X$ with $P\mathbb{k} = j \circ h$ there is a unique $\mathbb{h} : \mathbb{W} \to \mathbb{X}$ such that $\mathbb{j} \circ \mathbb{h} = \mathbb{k}$ and $P\mathbb{h} = h$. Moreover, $P : \mathcal{W} \to \mathcal{V}$ is called a (Grothendieck) fibration if all $j : X \to P\mathbb{C}$ have a cartesian lifting for all $\mathbb{C}$ in $\mathcal{W}$. If $P : \mathcal{W} \to \mathcal{V}$ is a fibration, the subcategory of $\mathcal{W}$ that has as arrows the arrows $\mathbb{f}$ such that $P\mathbb{f} = id_C$ is called the fibre over C.

The next lemma is a strengthening of Proposition 5.

▶ **Lemma 7.** Let $\mathcal{V}$ be a category with finite limits. The forgetful functor $\mathsf{Cat}(\mathcal{V}) \to \mathcal{V}$ is a fibration.

Instantiating Lemma 7 with $\mathbb{C} \times \mathbb{D}$ for $\mathbb{C}$ and $j : \mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$ for $j : X_0 \to \mathbb{C}_0$, gives us the desired result that internal categories can be pulled back along arbitrary arrows between objects-of-objects:

▶ **Corollary 8.** The arrow $j : \mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$ lifts to a morphism of internal categories $\mathbb{j} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{C} \times \mathbb{D}$. Moreover, $\mathbb{j}$ is the cartesian lifting of j.

To show that this construction is functorial we need to use that $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is functorial and that $j : \mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$ is natural in $\mathbb{C}$ and $\mathbb{D}$. In order to lift such natural transformations, which are arrows in the functor category $\mathcal{V}^{\mathsf{Cat}(\mathcal{V}) \times \mathsf{Cat}(\mathcal{V})}$, we use

▶ **Lemma 9.** If $P : \mathcal{E} \to \mathcal{B}$ is a fibration and $\mathcal{A}$ is a category, then $P^{\mathcal{A}} : \mathcal{E}^{\mathcal{A}} \to \mathcal{B}^{\mathcal{A}}$ is a fibration.

Instantiating the lemma with $P = (-)_0 : \mathsf{Cat}(\mathcal{V}) \to \mathcal{V}$ and $\mathcal{A} = \mathsf{Cat}(\mathcal{V}) \times \mathsf{Cat}(\mathcal{V})$, we obtain as a corollary that lifting the tensor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ to $\otimes : Cat(\mathcal{V}) \times Cat(\mathcal{V}) \to Cat(\mathcal{V})$ is functorial:

▶ **Theorem 10.** Let $(\mathcal{V}, 1, \otimes)$ be a (symmetric) monoidal category with finite limits in which the monoidal unit is the terminal object. Let $(-) : \mathsf{Cat}(\mathcal{V}) \to \mathcal{V}$ be the forgetful functor from categories internal in $\mathcal{V}$. Then the canonical arrow $j : \mathbb{C}_0 \otimes \mathbb{D}_0 \to \mathbb{C}_0 \times \mathbb{D}_0$ lifts to a natural transformation $\mathbb{j} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{C} \times \mathbb{D}$. Moreover, $(\mathsf{Cat}(\mathcal{V}), \mathbb{I}, \otimes)$ inherits from $(\mathcal{V}, 1, \otimes)$ the structure of a (symmetric) monoidal category with finite limits in which the monoidal unit is the terminal object.

In this paper we only need internal monoidal categories that are strict. In the same way as a strict monoidal category is a monoid in $(\mathsf{Cat}, \mathbb{1}, \times)$, an internal strict monoidal category is a monoid in $(\mathsf{Cat}(\mathcal{V}), \mathbb{I}, \otimes)$:

▶ **Definition 11** (Internal monoidal category)**.** Let $(\mathcal{V}, 1, \otimes)$ be a symmetric monoidal category with finite limits in which the monoidal unit is the terminal object and let $(\mathsf{Cat}(\mathcal{V}), \mathbb{I}, \otimes)$ be the induced symmetric monoidal category of internal categories in $\mathcal{V}$. A strict internal monoidal category $\mathbb{C}$ is a monoid $(\mathbb{C}, \varnothing, \odot)$ in $(\mathsf{Cat}(\mathcal{V}), \mathbb{I}, \otimes)$.

More explicitly, a strict internal monoidal category $\mathbb{C}$ has operations

$$\varnothing : \mathbb{I} \to \mathbb{C} \qquad \odot : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$$

satisfying the laws of a monoid. For example, in the category $\mathsf{n}\mathbb{F}$ of finite sets of names, $\varnothing$ is the empty set and $\uplus = \odot$ is, on objects, union of disjoint sets and, on arrows, union of functions with both disjoint domains and disjoint codomains. It follows from Remark 34 that an internal monoidal category satisfies the interchange law

$$\begin{array}{ccc} (\mathbb{C}\otimes\mathbb{C})_2 & \xrightarrow{\;comp\times comp\;} & (\mathbb{C}\otimes\mathbb{C})_1 \\ \downarrow{\scriptstyle \odot_2} & & \downarrow{\scriptstyle \odot_1} \\ \mathbb{C}_2 & \xrightarrow{\;comp\;} & \mathbb{C}_1 \end{array}$$

which can also be written as

$$(f \odot f') \,;(g \odot g') = (f; g) \odot (f'; g')$$

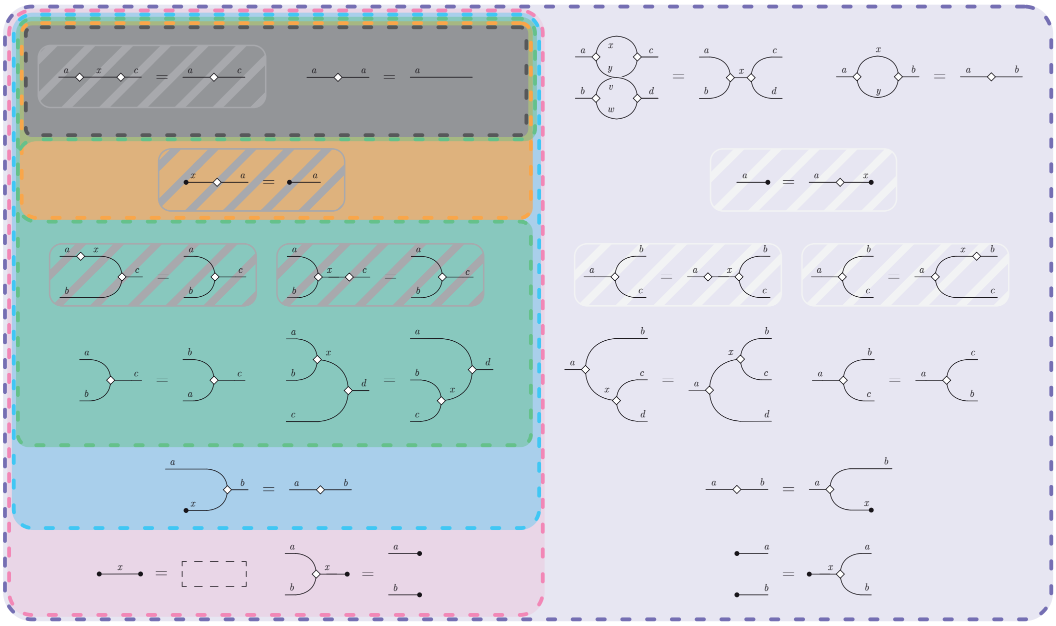
The move from $\times$ to $\otimes$ means that it is now possible that the right-hand side of the equation is defined while the left-hand side is not. But, as we will see, in nominal sets, the right-hand is always α-equivalent to one for which a left-hand side exists.

4 Examples

Before we give a formal definition of nominal PROPs and nominal monoidal theories (NMTs) in the next section, we present as examples those NMTs that correspond to the SMTs of Fig 3. The significant differences between Fig 3 and 4 are that wires now carry labels and that there is a new generator $\overset{a_i}{\longrightarrow}\!\diamond\!\overset{b_i}{\longrightarrow}$ which allows us to change the label of a wire. Moreover, in the nominal setting rules for wire crossings are not needed.

▶ **Theorem 12.** The calculi of Fig 4 are sound and complete, that is, the categories presented by these calculi are isomorphic the categories of finite sets of names with the respective maps.

The proof follows the same general lines as the well-known proofs for SMTs (see eg Lafont [14]) and proceed by showing that each diagram $f : A \to B$ can be rewritten to one in normal form, with the normal form being a direct syntactic representation of the semantic function/relation represented by f. The proofs for NMTs seem easier than the corresponding proofs for SMTs due to the absence of wire crossings. For example, in the case of bijections, it is immediate that, using the grey rules of Fig.4, every nominal diagram rewrites to a normal form which is just a parallel composition of diagrams of the form $\overset{a_i}{\longrightarrow}\!\diamond\!\overset{b_i}{\longrightarrow}$.



bijections n𝔹, injections n𝕀, surjections n𝕊, functions n𝔽, partial functions nℙ and relations nℝ

Figure 4 Nominal monoidal theories

5 Nominal monoidal theories and nominal PROPs

In this section, we introduce nominal PROPs as internal monoidal categories in nominal sets. We first spell out the details of what that means in elementary terms and then discuss the notion of diagrammatic alpha-equivalence.

5.1 Nominal monoidal theories

A *nominal monoidal theory* (Σ, E) is given by a nominal set Σ of generators and a nominal set E of equations. The set of nominal generators is itself generated by a set Σ_{o} of 'ordinary' generators $\gamma : n \to m$, each γ giving rise to a set of nominal generators $[\boldsymbol{a}\rangle\gamma\langle\boldsymbol{b}] : A \to B$ where $\boldsymbol{a}, \boldsymbol{b}$ are unique lists of size n, m and whose underlying sets are A, B respecitvely. The nominal generators Σ are closed under permutations

$$\pi \cdot [\boldsymbol{a}\rangle\gamma\langle\boldsymbol{b}] : \pi \cdot A \to \pi \cdot B = [\pi(\boldsymbol{a})\rangle\gamma\langle\pi(\boldsymbol{b})]. \qquad (\pi\text{-def})$$

The set of terms is given by closing under the operations of Fig 5, which should be compared with Fig 1.

Every NMT freely generates a monoidal category internal in nominal sets by quotienting the generated terms by:

- the equations that state that id and ; obey the laws of a category
- the equations stating that $id_{\varnothing}$ and $\uplus$ are a monoid
- the equations of an internal monoidal category of Fig 6 [3]

[3] The main difference with the equations in Fig 2 is that the interchange law for $\uplus$ is required to hold only

$$\frac{\gamma : m \to n \in \Sigma_{\mathsf{o}}}{[\boldsymbol{a}\rangle\gamma\langle\boldsymbol{b}] : A \to B} \qquad \frac{}{id_a : \{a\} \to \{a\}} \qquad \frac{}{\delta_{ab} : \{a\} \to \{b\}}$$

$$\frac{t : A \to B \qquad t' : A' \to B'}{t \uplus t' : A \uplus A' \to B \uplus B'} \qquad \frac{t : A \to B \qquad s : B \to C}{t\,;s : A \to C} \qquad \frac{t : A \to B}{(a\ b)\,t : (a\ b)\cdot A \to (a\ b)\cdot B}$$

Figure 5 NMT Terms

- the equations of permutation actions of Fig 7
- the equations on the interaction of generators with bijections δ of Fig 8
- the equations E

$$t \uplus s = s \uplus t \tag{NMT-comm}$$

$$(s\,;t) \uplus (u\,;v) = (s \uplus u)\,;(t \uplus v) \tag{NMT-ch}$$

Figure 6 NMT Equations of $\uplus$

$$(a\ b)id_x = id_{(a\ b)\cdot x} \qquad (a\ b)\delta_{xy} = \delta_{(a\ b)\cdot x\ (a\ b)\cdot y} \qquad (a\ b)\gamma = (a\ b)\cdot\gamma$$

$$(a\ b)(x \uplus y) = (a\ b)x \uplus (a\ b)y \qquad (a\ b)(x\,;y) = (a\ b)x\,;(a\ b)y$$

Figure 7 NMT Equations of the permutation actions

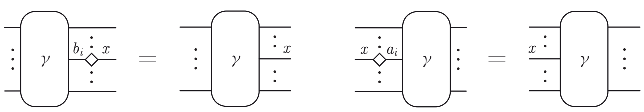
$$\delta_{ab}\,;\delta_{bc} = \delta_{ac}$$

$$\frac{[\boldsymbol{a}\rangle\gamma\langle b_1,\ldots,b_i,\ldots,b_n] : A \to B \uplus \{b_i\}}{[\boldsymbol{a}\rangle\gamma\langle b_1,\ldots,b_i,\ldots,b_n]\,;(id_B \uplus \delta_{b_i x}) = [\boldsymbol{a}\rangle\gamma\langle b_1,\ldots,x,\ldots,b_n]} \tag{NMT-right}$$

$$\frac{[a_1,\ldots,a_i,\ldots,a_m\rangle\gamma\langle\boldsymbol{b}] : \{a_i\} \uplus A \to B}{(\delta_{xa_i} \uplus id_A)\,;[a_1,\ldots,a_i,\ldots,a_m\rangle\gamma\langle\boldsymbol{b}] = [a_1,\ldots,x,\ldots,a_m\rangle\gamma\langle\boldsymbol{b}]} \tag{NMT-left}$$

Figure 8 NMT Equations of renamings

For terms to form a nominal set, we need equations between permutations (not listed here) to hold, as well as the equations of Fig 7 that specify how permutations act on terms. All the equations presented in the figures above are routine, with the possible exception of those of Fig 8, specifying the interaction of renamings δ with the generators $[\boldsymbol{a}\rangle\gamma\langle\boldsymbol{b}] \in \Sigma$, which we also depict in diagrammatic form:



if both sides are defined and that the two laws involving symmetries are replaced by the commutativity of $\uplus$.

Instances of these rules can be seen in Fig 4, where they are distinguished by a striped background.

5.2 Diagrammatic alpha-equivalence

The equations of Fig 7 and Fig 8 introduce a notion of *diagrammatic alpha-equivalence*, which allows us to rename 'internal' names and to contract renamings.

▶ **Definition 13.** Two terms of a nominal monoidal theory are alpha-equivalent if their equality follows from the equations in Fig 7 and Fig 8.

Every permutation π of names gives rise to bijective functions $\pi_A : A \to \pi[A] = \{\pi(a) \mid a \in A\} = \pi \cdot A$. Any such π_A, as well as the inverse π_A^{-1}, are parallel compositions of δ_{ab} for suitable $a, b \in \mathcal{N}$. In fact, we have $\pi_A = \biguplus_{a \in A} \delta_{a\pi(a)}$. We may therefore use the π_A as abbreviations in terms.

▶ **Proposition 14.** Let $t : A \to B$ be a term of a nominal monoidal theory. The equations in Fig 7 and Fig 8 entail that $\pi \cdot t = (\pi_A)^{-1}; t; \pi_B$.

$$\begin{array}{ccc} A & \xrightarrow{\quad t \quad} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ \pi[A] & \xrightarrow[\pi \cdot t]{\quad\quad} & \pi[B] \end{array}$$

The next two corollaries show that internal names can be renamed. We call this diagrammatic α-equivalence.

▶ **Corollary 15.** Let $t : A \uplus \{c\} \to B \uplus \{c\}$ be a term of a nominal monoidal theory and d be fresh for t. Then $t = (\delta_{cd} \uplus id_A); (c\ d) \cdot t; (\delta_{dc} \uplus id_B)$.

▶ **Corollary 16.** Let $t : A \to B$ be a term of a nominal monoidal theory. Modulo the equations of Fig 7 and Fig 8, the support of t is $A \cup B$.

The last corollary shows that internal names are bound by sequential composition. Indeed, in a composition $A \xrightarrow{t} C \xrightarrow{s} B$, the names in $C \setminus (A \cup B)$ do not appear in the support of $t; s$.

5.3 Nominal PROPs

From the point of view of Section 3, a nominal PROP is an internal strict monoidal category in $(\mathsf{Nom}, 1, *)$ that has finite sets of names as objects and at least all bijections as arrows. A functor between nominal PROPs is an internal functor that preserves objects and bijections. We spell this out in detail.

▶ Remark 17. A nominal PROP $\mathbb{C}$ is a small category, with a set $\mathbb{C}_0$ of 'objects' and a set $\mathbb{C}_1$ of 'arrows', defined as follows. We write ; for the sequential composition (in the diagrammatic order) and $\uplus$ for the monoidal composition.

- $\mathbb{C}_0$ is the set of finite subsets of a countably infinite set $\mathcal{N}$. The permutation action is given by $\pi \cdot A = \pi[A] = \{\pi(a) \mid a \in A\}$ for all finite permutations $\pi : \mathcal{N} \to \mathcal{N}$.
- $\mathbb{C}_1$ contains all bijections ('renamings') $\pi_A : A \to \pi \cdot A$, $\pi_A(a) = \pi(a)$, for all finite permutations $\pi : \mathcal{N} \to \mathcal{N}$ and is closed under the operation mapping an arrow $f : A \to B$ to $\pi \cdot f : \pi \cdot A \to \pi \cdot B$ defined as $\pi \cdot f = (\pi_A)^{-1};f;\pi_B$.

- $A \uplus B$ is the union of A and B and defined whenever A and B are disjoint. This makes $(\mathbb{C}_0, \emptyset, \uplus)$ a commutative partial monoid. On arrows, we require $(\mathbb{C}_1, \emptyset, \uplus)$ to be a commutative partial monoid, with $f \uplus g$ defined whenever $dom f \cap dom g = \emptyset$ and $cod f \cap cod g = \emptyset$.
- The interchange law $(f \uplus f');(g \uplus g') = (f\,;g) \uplus (f'\,;g')$ holds whenever the left-hand side is defined.

From this definition on can deduce the following.

▶ *Remark* 18.

- A nominal PROP has a nominal set of objects and a nominal set of arrows.
- The support of an object A is A and the support of an arrow $f : A \to B$ is $A \cup B$. In particular, $\mathsf{supp}(f;g) = dom(f) \cup cod(g)$. In other words, nominal PROPs have diagrammatic alpha equivalence.
- There is a category nPROP that consists of nominal PROPs together with functors that are the identity on objects and bijections and are strict monoidal and equivariant.
- Every NMT presents a nPROP. Conversely, every nPROP is presented by at least one NMT given by all terms as generators and all equations.

6 Equivalence of nominal and ordinary string diagrams

We show that the categories nPROP and PROP are equivalent. To define translations between ordinary and nominal monoidal theories we introduce some auxiliary notation. We denote lists that contain each letter at most once by bold letters. If $\boldsymbol{a} = [a_1, \dots a_n]$ is a list, then $\underline{\boldsymbol{a}} = \{a_1, \dots a_n\}$. Given lists $\boldsymbol{a}$ and $\boldsymbol{a'}$ with $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a'}}$ we abbreviate bijections in PROP (also called symmetries) mapping $i \mapsto a_i = a'_j \mapsto j$ as $\langle \boldsymbol{a} | \boldsymbol{a'} \rangle$. Given lists $\boldsymbol{a}$ and $\boldsymbol{b}$ of the same length we write $[\boldsymbol{a}|\boldsymbol{b}] = \biguplus \delta_{a_i b_i}$ for the bijection $a_i \mapsto b_i$ in an nPROP.

▶ **Proposition 19.** For any PROP $\mathcal{S}$, there is an nPROP

$$NOM(\mathcal{S})$$

that has for all arrows $f : \underline{n} \to \underline{m}$ of $\mathcal{S}$, and for all lists $\boldsymbol{a} = [a_1, \dots a_n]$ and $\boldsymbol{b} = [b_1, \dots b_m]$ arrows $[\boldsymbol{a}\rangle f \langle \boldsymbol{b}] \in NOM(\mathcal{S})$. These arrows are subject to equations

$$[\boldsymbol{a}\rangle f\,; g\langle \boldsymbol{c}] = [\boldsymbol{a}\rangle f\langle \boldsymbol{b}]\,; [\boldsymbol{b}\rangle g\langle \boldsymbol{c}] \tag{NOM-1}$$
$$[\boldsymbol{a} + \boldsymbol{c}\rangle f \oplus g\langle \boldsymbol{b} + \boldsymbol{d}] = [\boldsymbol{a}\rangle f\langle \boldsymbol{b}] \uplus [\boldsymbol{c}\rangle g\langle \boldsymbol{d}] \tag{NOM-2}$$
$$[\boldsymbol{a}\rangle id\langle \boldsymbol{b}] = [\boldsymbol{a}|\boldsymbol{b}] \tag{NOM-3}$$
$$[\boldsymbol{a}\rangle\, \langle \boldsymbol{b}|\boldsymbol{b'}\rangle\,; f\,\langle \boldsymbol{c}] = [\boldsymbol{a}|\boldsymbol{b}]\,; [\boldsymbol{b'}\rangle f\langle \boldsymbol{c}] \tag{NOM-4}$$
$$[\boldsymbol{a}\rangle\, f\,; \langle \boldsymbol{b}|\boldsymbol{b'}\rangle\,\langle \boldsymbol{c}] = [\boldsymbol{a}\rangle f\langle \boldsymbol{b}]\,; [\boldsymbol{b'}|\boldsymbol{c}] \tag{NOM-5}$$

Proof. To show that $NOM(\mathcal{S})$ is well-defined, we need to check that the equations of $\mathcal{S}$ are respected. We only have space here for the most interesting case which is the naturality of symmetries given by the last equation in Fig 2. We write $\boldsymbol{a}^m$ for a list of a's of length m.

$$
\begin{aligned}
& [\boldsymbol{a}^m + \boldsymbol{a}^z\rangle\,(t \oplus id_z)\,; \sigma_{n,z}\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] \\
&= ([\boldsymbol{a}^m\rangle\, t\,\langle \boldsymbol{x}^n] \uplus [\boldsymbol{a}^z\rangle\, id_z\,\langle \boldsymbol{x}^z]) ; [\boldsymbol{x}^n + \boldsymbol{x}^z\rangle\,\sigma_{n,z}\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && \text{(NOM-1,2)} \\
&= ([\boldsymbol{a}^z\rangle\, id_z\,\langle \boldsymbol{x}^z] \uplus [\boldsymbol{a}^m\rangle\, t\,\langle \boldsymbol{x}^n]) ; [\boldsymbol{x}^n + \boldsymbol{x}^z\rangle\,\sigma_{n,z}\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && \text{(NMT-comm)} \\
&= [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{x}^z + \boldsymbol{x}^n] ; [\boldsymbol{x}^n + \boldsymbol{x}^z\rangle\,\sigma_{n,z}\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && \text{(NOM-2)} \\
&= [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{x}^z + \boldsymbol{x}^n] ; [\boldsymbol{x}^n + \boldsymbol{x}^z\rangle\,\langle \boldsymbol{x}^n + \boldsymbol{x}^z | \boldsymbol{x}^z + \boldsymbol{x}^n\rangle\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && (\sigma\text{-def}) \\
&= [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{x}^z + \boldsymbol{x}^n] ; [\boldsymbol{x}^n + \boldsymbol{x}^z | \boldsymbol{x}^n + \boldsymbol{x}^z] ; [\boldsymbol{x}^z + \boldsymbol{x}^n | \boldsymbol{b}^z + \boldsymbol{b}^n] \\
&= [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{x}^z + \boldsymbol{x}^n] ; [\boldsymbol{x}^z + \boldsymbol{x}^n | \boldsymbol{b}^z + \boldsymbol{b}^n] && (\delta_{aa} = id_a) \\
&= [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && \text{(NOM-5)} \\
&= [\boldsymbol{a}^m + \boldsymbol{a}^z | \boldsymbol{a}^m + \boldsymbol{a}^z] \;;\; [\boldsymbol{a}^z + \boldsymbol{a}^m\rangle\, id_z \oplus t\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && (\delta_{aa} = id_a) \\
&= [\boldsymbol{a}^m + \boldsymbol{a}^z\rangle\,\langle \boldsymbol{a}^m + \boldsymbol{a}^z | \boldsymbol{a}^z + \boldsymbol{a}^m\rangle ; (id_z \oplus t)\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && \text{(NOM-4)} \\
&= [\boldsymbol{a}^m + \boldsymbol{a}^z\rangle\,\sigma_{m,z} ; (id_z \oplus t)\,\langle \boldsymbol{b}^z + \boldsymbol{b}^n] && (\sigma\text{-def})
\end{aligned}
$$

Note how commutativity of $\uplus$ is used to show that naturality of symmetries is respected. ◀

▶ **Proposition 20.** For any nPROP $\mathcal{T}$ there is a PROP

$$ORD(\mathcal{T})$$

that has for all arrows $f : A \to B$ of $\mathcal{T}$, and for all lists $\boldsymbol{a} = [a_1, \dots a_n]$ and $\boldsymbol{b} = [b_1, \dots b_m]$ arrows $\langle \boldsymbol{a}] f [\boldsymbol{b}\rangle$. These arrows are subject to equations

$$
\begin{aligned}
\langle \boldsymbol{a}]\, f ; g\, [\boldsymbol{c}\rangle &= \langle \boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle ; \langle \boldsymbol{b}]\, g\, [\boldsymbol{c}\rangle && \text{(ORD-1)} \\
\langle \boldsymbol{a}_f + \boldsymbol{a}_g]\, f \uplus g\, [\boldsymbol{b}_f + \boldsymbol{b}_g\rangle &= \langle \boldsymbol{a}_f]\, f\, [\boldsymbol{b}_f\rangle \oplus \langle \boldsymbol{a}_g]\, g\, [\boldsymbol{b}_g\rangle && \text{(ORD-2)} \\
\langle \boldsymbol{a}]\, id\, [\boldsymbol{a}\rangle &= id && \text{(ORD-3)} \\
\langle \boldsymbol{a}]\, [\boldsymbol{a}' | \boldsymbol{b}] ; f\, [\boldsymbol{c}\rangle &= \langle \boldsymbol{a} | \boldsymbol{a}'\rangle ; \langle \boldsymbol{b}]\, f\, [\boldsymbol{c}\rangle && \text{(ORD-4)} \\
\langle \boldsymbol{a}]\, f ; [\boldsymbol{b} | \boldsymbol{c}]\, [\boldsymbol{c}'\rangle &= \langle \boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle ; \langle \boldsymbol{c} | \boldsymbol{c}'\rangle && \text{(ORD-5)}
\end{aligned}
$$

Proof. To show that *ORD* is well-defined we need to show that the equations of an NMT are respected. The most interesting case here is the commutativity of $\uplus$ since the $\oplus$ of SMTs is not commutative.

$$
\begin{aligned}
& \langle \boldsymbol{a}_t + \boldsymbol{a}_s]\, t \uplus s\, [\boldsymbol{b}_t + \boldsymbol{b}_s\rangle \\
&= \langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle \oplus \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle && \text{(ORD-2)} \\
&= (\langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle \;;\; id_{|\boldsymbol{b}_t|}) \oplus (id_{|\boldsymbol{a}_s|} \;;\; \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle) && (id ; a = a = a ; id) \\
&= (\langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle \oplus id_{|\boldsymbol{a}_s|}) \;;\; (id_{|\boldsymbol{b}_t|} \oplus \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle) && \text{(SMT-ch)} \\
&= (\langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle \oplus id_{|\boldsymbol{a}_s|}) \;;\; \sigma_{|\boldsymbol{b}_t|,|\boldsymbol{a}_s|} \;;\; \sigma_{|\boldsymbol{a}_s|,|\boldsymbol{b}_t|} \;;\; (id_{|\boldsymbol{b}_t|} \oplus \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle) && \text{(SMT-sym)} \\
&= \sigma_{|\boldsymbol{a}_t|,|\boldsymbol{a}_s|} \;;\; (id_{|\boldsymbol{a}_s|} \oplus \langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle) \;;\; \sigma_{|\boldsymbol{a}_s|,|\boldsymbol{b}_t|} \;;\; (id_{|\boldsymbol{b}_t|} \oplus \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle) && \text{(SMT-nat)} \\
&= \sigma_{|\boldsymbol{a}_t|,|\boldsymbol{a}_s|} \;;\; (id_{|\boldsymbol{a}_s|} \oplus \langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle) \;;\; (\langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle \oplus id_{|\boldsymbol{b}_t|}) \;;\; \sigma_{|\boldsymbol{b}_s|,|\boldsymbol{b}_t|} && \text{(SMT-nat)} \\
&= \sigma_{|\boldsymbol{a}_t|,|\boldsymbol{a}_s|} \;; ((id_{|\boldsymbol{a}_s|} \;;\; \langle \boldsymbol{a}_s]\, s\, [\boldsymbol{b}_s\rangle) \oplus (\langle \boldsymbol{a}_t]\, t\, [\boldsymbol{b}_t\rangle \;;\; id_{|\boldsymbol{b}_t|})) \;;\; \sigma_{|\boldsymbol{b}_s|,|\boldsymbol{b}_t|} && \text{(SMT-ch)} \\
&= \sigma_{|\boldsymbol{a}_t|,|\boldsymbol{a}_s|} \;;\; \langle \boldsymbol{a}_s + \boldsymbol{a}_t]\, s \uplus t\, [\boldsymbol{b}_s + \boldsymbol{b}_t\rangle \;;\; \sigma_{|\boldsymbol{b}_s|,|\boldsymbol{b}_t|} && (id ; a = a, \text{ORD-2}) \\
&= \langle \boldsymbol{a}_t + \boldsymbol{a}_s | \boldsymbol{a}_s + \boldsymbol{a}_t\rangle \;;\; \langle \boldsymbol{a}_s + \boldsymbol{a}_t]\, s \uplus t\, [\boldsymbol{b}_s + \boldsymbol{b}_t\rangle \;;\; \langle \boldsymbol{b}_s + \boldsymbol{b}_t | \boldsymbol{b}_t + \boldsymbol{b}_s\rangle && (\sigma\text{-def}) \\
&= \langle \boldsymbol{a}_t + \boldsymbol{a}_s]\, [\boldsymbol{a}_s + \boldsymbol{a}_t | \boldsymbol{a}_s + \boldsymbol{a}_t] \;;\; s \uplus t \;;\; [\boldsymbol{b}_s + \boldsymbol{b}_t | \boldsymbol{b}_s + \boldsymbol{b}_t]\, [\boldsymbol{b}_t + \boldsymbol{b}_s\rangle && \text{(ORD-4,5)} \\
&= \langle \boldsymbol{a}_t + \boldsymbol{a}_s]\, s \uplus t\, [\boldsymbol{b}_t + \boldsymbol{b}_s\rangle && (\delta_{aa} = id_a)
\end{aligned}
$$

Note how naturality of symmetries is used to show that the definition of ORD respects commutativity of $\uplus$. ◀

Having described the maps NOM and ORD and shown they are homomorphisms, we now describe functors $NOM(F)$ and $ORD(F)$.

▶ **Proposition 21.** NOM : PROP $\to$ nPROP is a functor mapping an arrow of PROPs $F : \mathcal{S} \to \mathcal{S}$ to an arrow of nPROPs $NOM(F) : NOM(\mathcal{S}) \to NOM(\mathcal{S})$ defined by

$$NOM(F)([\boldsymbol{a}\rangle\, g\, \langle\boldsymbol{b}]) = [\boldsymbol{a}\rangle\, Fg\, \langle\boldsymbol{b}]. \tag{NOM-F}$$

▶ **Proposition 22.** ORD is a functor mapping an arrow of nPROPs $F : \mathcal{T} \to \mathcal{T}$ to an arrow of PROPs $ORD(F) : ORD(\mathcal{T}) \to ORD(\mathcal{T})$ defined by

$$ORD(F)(\langle\boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle) = \langle\boldsymbol{a}]\, Ff\, [\boldsymbol{b}\rangle \tag{ORD-F}$$

The next proposition has a variation in which we take PROPs in the weaker sense of Lack [13]. Then the unit $\mathcal{S} \to ORD(NOM(\mathcal{S}))$ is not an iso. To see where we need to be careful, the next example illustrates how the commutativity of $\uplus$ in an nPROP translates into the naturality of the symmetries in a PROP.

▶ **Example 23** (Commutativity of $\uplus$ translates to naturality of symmetries)**.** If $\mathcal{S}$ is a PROP in the sense of Lack [13] generated by a 'lollipop' $\lambda : 0 \to 1$ then we can show that $\lambda \oplus id$ and $(id \oplus \lambda)\,; \sigma_{1,1}$ in $\mathcal{S}$ are sent to the same arrow in $ORD(NOM(\mathcal{S}))$, namely we can show $\langle a][a\rangle\lambda \oplus id\langle b,c][b,c\rangle = \langle a][a\rangle(id \oplus \lambda)\,; \sigma_{1,1}\langle b,c][b,c\rangle$:

$$\begin{aligned}
\langle a][a\rangle\lambda \oplus id\langle b,c][b,c\rangle &= \langle a][\,\rangle\lambda\langle b] \uplus [a\rangle id\langle c][b,c\rangle && \text{(NOM-2)}\\
&= \langle a][a\rangle id\langle c] \uplus [\,\rangle\lambda\langle b][b,c\rangle && \text{(NMT-comm)}\\
&= \langle a][a\rangle id \oplus \lambda\langle c,b][b,c\rangle && \text{(NOM-2)}\\
&= \langle a][a\rangle id \oplus \lambda\langle c,b]\,; [b,c|b,c][b,c\rangle && (a = a\,; id, \delta_{aa} = id_a)\\
&= \langle a][a\rangle(id \oplus \lambda)\,; \langle c,b|b,c\rangle\langle b,c][b,c\rangle && \text{(NOM-5)}\\
&= \langle a][a\rangle(id \oplus \lambda)\,; \sigma_{1,1}\langle b,c][b,c\rangle && (\sigma\text{-def})
\end{aligned}$$

which is an instance of (SMT-nat) and does not hold in $\mathcal{S}$.

As we can see from the example, the naturality of symmetries in a PROP is necessary in order to obtain that $\mathcal{S} \to ORD(NOM(\mathcal{S}))$ is an iso in the next proposition.

▶ **Proposition 24.** For each PROP $\mathcal{S}$, there is an isomorphism of PROPs, natural in $\mathcal{S}$,

$$\Delta : \mathcal{S} \to ORD(NOM(\mathcal{S}))$$

mapping $f \in \mathcal{S}$ to $\langle\boldsymbol{a}][\boldsymbol{a}\rangle\, f\, \langle\boldsymbol{b}][\boldsymbol{b}\rangle$ for some choice of $\boldsymbol{a}, \boldsymbol{b}$.

▶ **Proposition 25.** For each nPROP $\mathcal{T}$, there is an isomorphism of nPROPs, natural in $\mathcal{T}$,

$$NOM(ORD(\mathcal{T})) \to \mathcal{T}$$

mapping the $[\boldsymbol{c}\rangle\langle\boldsymbol{a}]\, f\, [\boldsymbol{b}\rangle\langle\boldsymbol{d}]$ generated by an $f : \underline{\boldsymbol{a}} \to \underline{\boldsymbol{b}}$ in $\mathcal{T}$ to $[\boldsymbol{c}|\boldsymbol{a}]\,; f\,; [\boldsymbol{b}|\boldsymbol{d}]$.

Since the last two propositions provide an isomorphic unit and counit of an adjunction, we obtain

▶ **Theorem 26.** The categories PROP and nPROP are equivalent.

▶ Remark 27. If we generalise the notion of PROP from MacLane [15] to Lack [13], in other words, if we drop equation (SMT-nat) of Fig 2 expressing the naturality of symmetries, we still obtain an adjunction, in which NOM is left-adjoint to ORD. Nominal PROPs then are a full reflective subcategory of ordinary PROPs. In other words, the (generalised) PROPs that satisfy naturality of symmetries are exactly those which are nominal PROPs.

7 Conclusion

The equivalence of nominal and ordinary PROPs (Theorem 26) has a satisfactory graphical interpretation. Indeed, comparing Figs 3 and 4 we see that both share, modulo different labellings of wires mediated by the functors ORD and NOM, the same core of generators and equations while the main difference lies in the equations expressing, on the one hand, that $\oplus$ has natural symmetries and, on the other hand, that generators are a nominal set and $\uplus$ is commutative. In fact, this can be taken as a justification of the importance of naturality, which, informally speaking, compensates for the irrelevant detail induced by ordering names.

There are several directions for future research. First, the notion of an internal monoidal category has been developed because it is easier to prove the basic results in general rather than only in the special case of nominal sets. Nevertheless, it would be interesting to explore whether there are other interesting instances of internal monoidal categories.

Second, internal monoidal categories are a principled way to build monoidal categories with a partial tensor. For example, by working internally in the category of nominal sets with the separated product we can capture in a natural way constraints such as the tensor $f \oplus g$ for two partial maps $f, g : \mathcal{N} \to V$ being defined only if the domains of f and g are disjoint. This reminds us of the work initiated by O'Hearn and Pym on categorical and algebraic models for separation logic and other resource logics, see eg [17, 9, 6]. It seems promising to investigate how to build categorical models for resource logics based on internal monoidal theories. In one direction, one could extend the work of Curien and Mimram [5] to partial monoidal categories. Another question is whether there is a more general strictification result characterising when a symmetric tensor can be replaced by a partial but commutative one.

Third, there has been substantial progress in exploiting Lack's work on composing PROPs [13] in order to develop novel string diagrammatic calculi for a wide range of applications, see eg [1, 2, 3, 22]. It will be interesting to explore how much of this technology can be transferred from PROPs to nominal PROPs.

Fourth, various applications of nominal string diagrams could be of interest. The original motivation for our work was to obtain a convenient calculus for simultaneous substitutions that can be integrated with multi-type display calculi [7] and, in particular, with the multi-type display calculus for first-order logic of Tzimoulis [21]. Another direction for applications comes from the work of Ghica and Lopez [10] on a nominal syntax for string diagrams. In particular, it would be of interest to add various binding operations to nominal PROPs.

References

1 John C. Baez, Brandon Coya, and Franciscus Rebro. Props in network theory. *Theory and Applications of Categories*, 33, 2018. URL: `http://www.tac.mta.ca/tac/volumes/33/25/33-25abs.html`.

2 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. Rewriting modulo symmetric monoidal structure. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, pages 710–719, New York, NY, USA, 2016. ACM. doi:10.1145/2933575.2935316.

3 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. The calculus of signal flow diagrams i. *Inf. Comput.*, 252(C):2–29, February 2017. doi:10.1016/j.ic.2016.03.002.

4 Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2017. doi:10.1017/9781316219317.

5 Pierre-Louis Curien and Samuel Mimram. Coherent Presentations of Monoidal Categories. *Logical Methods in Computer Science*, 13(3):1–38, September 2017. doi:10.23638/LMCS-13(3:31)2017.

6 Brijesh Dongol, Victor B. F. Gomes, and Georg Struth. A program construction and verification tool for separation logic. In *Mathematics of Program Construction - 12th International Conference, MPC 2015, Königswinter, Germany, June 29 - July 1, 2015. Proceedings*, pages 137–158, 2015. doi:10.1007/978-3-319-19797-5_7.

7 Sabine Frittella, Giuseppe Greco, Alexander Kurz, Alessandra Palmigiano, and Vlasta Sikimić. Multi-type display calculus for dynamic epistemic logic. *Journal of Logic and Computation*, 26(6):2017–2065, 12 2014. doi:10.1093/logcom/exu068.

8 Murdoch J. Gabbay and Andrew M. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13(3):341–363, Jul 2002. doi:10.1007/s001650200016.

9 D. Galmiche, D. Méry, and D. Pym. The semantics of bi and resource tableaux. *Mathematical. Structures in Comp. Sci.*, 15(6):1033–1088, December 2005. doi:10.1017/S0960129505004858.

10 Dan R. Ghica and Aliaume Lopez. A structural and nominal syntax for diagrams. In *Proceedings 14th International Conference on Quantum Physics and Logic, QPL 2017, Nijmegen, The Netherlands, 3-7 July 2017.*, pages 71–83, 2017. doi:10.4204/EPTCS.266.4.

11 Bart Jacobs. *Categorical Logic and Type Theory*. Studies in logic and the foundations of mathematics. Elsevier Science, 1999.

12 André Joyal and Ross Street. The geometry of tensor calculus, i. *Advances in Mathematics*, 88(1):55–112, 1991. doi:10.1016/0001-8708(91)90003-P.

13 Steve Lack. Composing props. *Theory and Applications of Categories*, 13:147–163, 2004. URL: http://www.tac.mta.ca/tac/volumes/13/9/13-09abs.html.

14 Yves Lafont. Towards an algebraic theory of boolean circuits. *Journal of Pure and Applied Algebra*, 184:257–310, 2003. doi:10.1016/S0022-4049(03)00069-0.

15 Saunders Mac Lane. *Categories for the Working Mathematician*. Springer New York, 1978. doi:10.1007/978-1-4757-4721-8.

16 Saunders MacLane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71(1):40–106, 01 1965. doi:10.1090/S0002-9904-1965-11234-4.

17 Peter W. O'Hearn and David J. Pym. The logic of bunched implications. *The Bulletin of Symbolic Logic*, 5(2):215–244, 1999. doi:10.2307/421090.

18 Andrew M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2013. doi:10.1017/CBO9781139084673.

19 P. Selinger. *A Survey of Graphical Languages for Monoidal Categories*, pages 289–355. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. doi:10.1007/978-3-642-12821-9_4.

20 Thomas Streicher. Fibred categories à la jean bénabou, 1999. arXiv:arXiv:1801.02927.

21 Apostolos Tzimoulis. *Algebraic and Proof-Theoretic Foundations of the Logics for Social Behaviour*. PhD thesis, Delft University of Technology, 2018. doi:10.4233/uuid:e67e7724-b378-4ca3-ad4e-c40df245af5e.

22 Fabio Zanasi. *Interacting Hopf Algebras- the Theory of Linear Systems*. Theses, Ecole normale supérieure de lyon - ENS LYON, October 2015. URL: https://tel.archives-ouvertes.fr/tel-01218015.

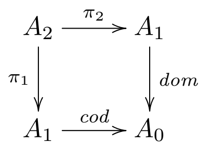
A Some internal category theory

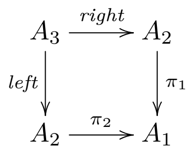
See eg Borceux, Handbook of Categorical Algebra, Volume 1, Chapter 8 and the nlab.

▶ **Definition 28** (internal category). *In a category with finite limits an* internal category *is a diagram*

$$A_3 \begin{array}{c} \xrightarrow{right} \\ \xrightarrow{compr} \\ \xrightarrow{compl} \\ \xrightarrow{left} \end{array} A_2 \begin{array}{c} \xrightarrow{\pi_2} \\ \xrightarrow{comp} \\ \xrightarrow{\pi_1} \end{array} A_1 \begin{array}{c} \xrightarrow{dom} \\ \xleftarrow{i} \\ \xrightarrow{cod} \end{array} A_0 \tag{5}$$

such that

1. *the 'pairs of arrows'-object A_2*  *is a pullback,*
2. *the 'triple of arrows'-object A_3 is a pullback*

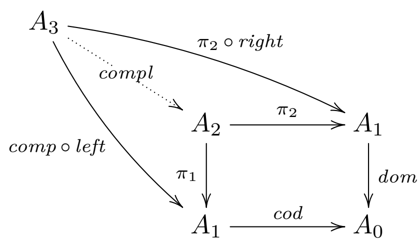


where, intuitively, left "projects out the left two arrows" and right "projects out the right two arrows"

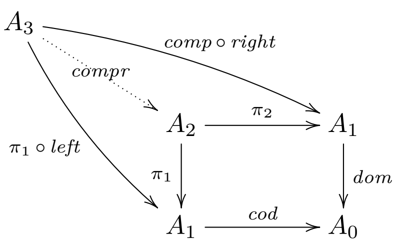
3. $dom \circ comp = dom \circ \pi_1$ *and* $cod \circ comp = cod \circ \pi_2$,
4. $dom \circ i = id_{A_0} = cod \circ i$,
5. $comp \circ \langle i \circ dom, id_{A_1} \rangle = id_{A_1} = comp \circ \langle id_{A_1}, i \circ cod \rangle$
6. $comp \circ compl = comp \circ compr$

where we use the auxiliary notation

- $\langle i \circ dom, id_{A_1} \rangle : A_1 \to A_2$ *and* $\langle id_{A_1}, i \circ cod \rangle : A_1 \to A_2$ *are the arrows into the pullback* A_2 *pairing* $i \circ dom, id_{A_1} : A_1 \to A_1$ *and* $id_{A_1}, i \circ cod : A_1 \to A_1$*, respectively,*
- *compl is the arrow composing the "left two arrows"*

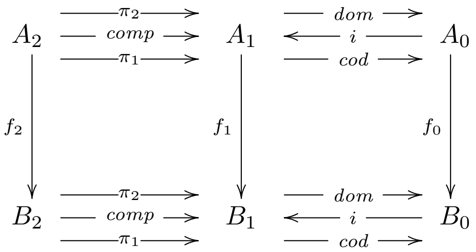


- *compr is the arrow composing the "right two arrows"*



▶ Remark 29. 1. and 3. define A_2 as the 'object of composable pairs of arrows' while 4. and 5. express that the 'object of arrows' A_1 has identities. 2. and 5. formalise associativity of composition. Since A_2 and A_3 are pullbacks, the structure is determined by $(A_0, A_1, \mathit{dom}, \mathit{cod}, i, \mathit{comp})$ alone. We included A_2, A_3 as well as $\mathit{compr}, \mathit{compl}, \mathit{right}, \mathit{left}, \pi_2, \pi_1$ to improve readability of the equations.

▶ **Definition 30.** *A morphism $f : A \to B$ between internal categories, an* internal functor*, is a pair (f_0, f_1) of arrows such that the six squares (one for each of π_2, comp, π_1, dom, cod, i)*



(6)

commute.

▶ Remark 31.
- Because B_2 is a pullback f_2 is uniquely determined by f_1. In more detail, if $\Gamma \to B_2$ is any arrow then, because B_2 is a pullback, it can be written as a pair

$$\langle l, r\rangle : \Gamma \to B_2 \tag{7}$$

  of arrows $l, r : \Gamma \to B_1$ and f_2 is determined by f_1 via

$$f_2 \circ \langle l, r\rangle = \langle f_1 \circ l, f_1 \circ r\rangle \tag{8}$$

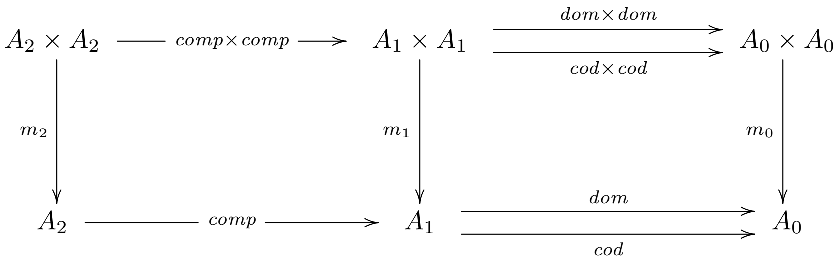
- Even if f_2 is not needed as part of the structure in the above definition, including f_2 makes it easier to state that f_1 preserves composition.
- Similarly, B_3 is a pullback, and there is a unique arrow f_3 such that (f_0, f_1, f_2, f_3) together make further 4 squares commute, one for each of *right*, *compr*, *compl*, *left*, see (5). We may include f_3 in the structure whenever convenient.

▶ **Definition 32.** *A natural transformation $\alpha : f \to g$ between internal functors $f, g : A \to B$, an* internal natural transformation*, is an arrow $\alpha : A_0 \to B_1$ such that, recalling* (7)*,*

$$\mathit{dom} \circ \alpha = f_0 \qquad \mathit{cod} \circ \alpha = g_0 \qquad \mathit{comp} \circ \langle f_1, \alpha \circ \mathit{cod}\rangle = \mathit{comp} \circ \langle \alpha \circ \mathit{dom}, g_1\rangle$$

▶ Remark 33. Internal categories with functors and natural transformations form a 2-category. We denote by $\mathsf{Cat}(\mathcal{V})$ the category or 2-category of categories internal in $\mathcal{V}$. The forgetful functor $\mathsf{Cat}(\mathcal{V}) \to \mathcal{C}$ mapping an internal category A to its object of objects A_0 has both left and right adjoints and, therefore, preserves limits and colimits. Moreover, a limit of internal categories is computed component-wise as $(\lim D)_j = \lim(D_j)$ for $j = 0, 1, 2$.

▶ Remark 34. A strict monoidal category can be thought of both as a monoid in the category of categories and as a category internal in the category of monoids. To understand this in more detail, note that both cases give rise to the diagram



where

- in the case of a monoid A in the category of internal categories, $m = (m_0, m_1, m_2)$ is an internal functor $A \times A \to A$ and, using that products of internal categories are computed component-wise, we have $comp \circ m_2 = m_1 \circ (comp \times comp)$, which gives us the interchange law

$$(f;g) \cdot (f';g') = (f \cdot f')\,;(g \cdot g')$$

  by using (8) with m for f and writing ; for *comp* and $\cdot$ for m_1;
- in the case of a category internal in monoids we have monoids A_0, A_1, A_2 and monoid homomorphisms i, *dom*, *cod*, *comp* which, if spelled out, leads to the same commuting diagrams as the previous item.